Euclidian Norm, Euclidian Distance, and Angle

Linear Algebra

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu

Maryam Ramezani maryam.ramezani@sharif.edu

Overview

- Introduction
- Inequalities
- Euclidean Norm
- Euclidean Metric (Distance)
- Angle

Introduction

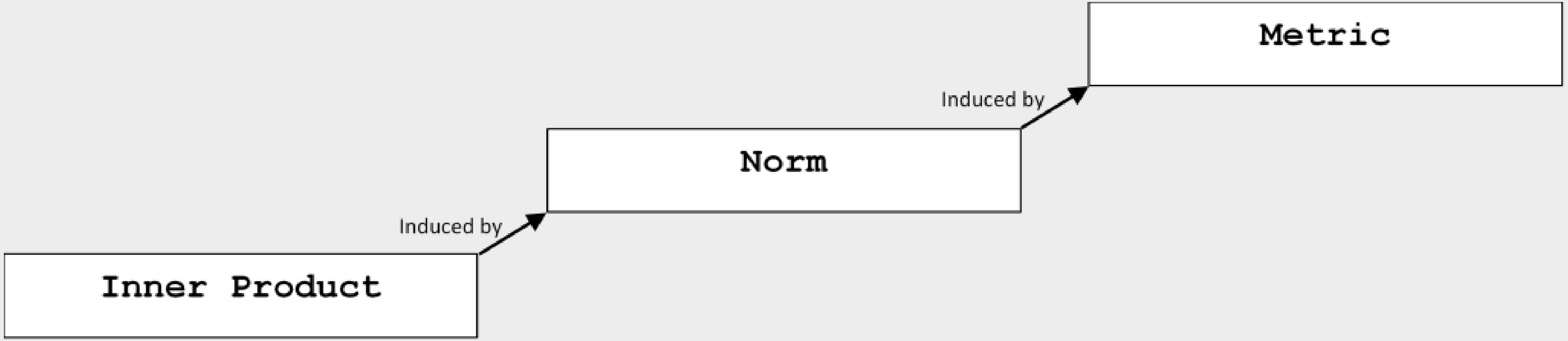
The reason to use norms

- Machine learning uses vectors, matrices, and tensors as the basic units of representation
- Two reasons to use norms:

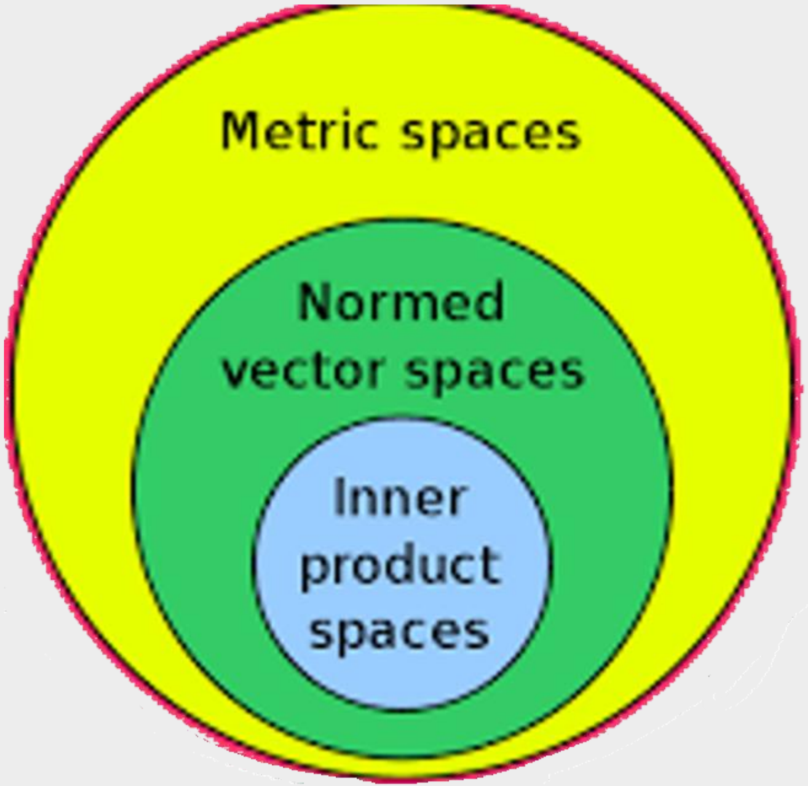
1. To estimate how **big** a vector/matrix/tensor is
   - How big is the difference between two tensors is

2. To estimate how **close** one tensor is to another
   - How close is one image to another

Inner Products, Norms and Metrics



- Given an inner product <A , B>, one can obtain a norm doing $|| A ||^2 = \langle A , A \rangle$
- And given a norm $|| A ||$, one can obtain a metric using the difference vector $|| A - B||$



Vector space	Generalization
metric	metric space
norm	normed
scalar product	inner product space

Definition

An inner product <,> , also called dot product, is a function that enables us to define and apply geometrical terms such as length, distance and angle in an Euclidean (vector) space .

Let V be a vector space over $\mathbb{R}$. An **inner product** on V is a function $\langle\ ,\rangle : V \times V \to \mathbb{R}$ such that for all $u, v, w \in V$ and $a, b \in \mathbb{R}$, the following hold:

1. $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0 \Leftrightarrow v = 0$;
2. $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$;
3. $\langle u, v\rangle = \langle v, u\rangle$.

https://youtu.be/LyGKycYT2v0

Norm

Definition

Functions closely related to inner products are so-called **norms**. Norms are specific functions that can be interpreted as a distance function between a vector and the origin.

Definition

For $v \in V$, we define the norm of v, denoted $||v||$, by:

$$||v|| = \sqrt{< v, v >}$$

Example

Norm of $P_n(x)$ in the term of inner product $\langle p_n(x), q_n(x) \rangle = \int_0^1 p_n(x) q_n(x) dx$:

$$||P_n(x)|| = \sqrt{\int_0^1 P_n^2(x) dx}$$

Definition

- Euclidean Norm (2-norm, l_2 norm, length)

$$||x|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{x^T x}$$

  - A vector whose length is 1 is called a unit vector
  - **Normalizing**: divide a non-zero vector by its length which is a unit vector in the same direction of original vector
  - It is a nonnegative scalar
  - In $\mathbb{R}^2$ follows from the Pythagorean Theorem.
  - What about $\mathbb{R}^3$?
  - What is the shape of $||x||_2 = 1$?

Inequalities

Chebyshev Inequality

Theorem

Suppose that k of the numbers $|x_1|, |x_2|, \dots, |x_n|$ are $\geq a$ then k of the numbers $x_1^2, x_2^2, \dots, x_n^2\ are \geq a^2$

So $||x||^2 = x_1^2 + x_2^2 + \ \dots + x_n^2 \geq ka^2\ so\ we\ have\ k \leq \frac{||x||^2}{a^2}$

Number of x_i with $|x_i| \geq a$ is no more than $\frac{||x||^2}{a^2}$

Question

- What happens when $\frac{||x||^2}{a^2} \geq n$?
- No entry of a vector can be larger in magnitude than the norm of the vector. (why?)

Cauchy–Schwartz Inequality

Theorem

For two n–vectors a and b, $|a^T b| \leq \; ||a|| \, ||b||$

Written out:

$$|a_1 b_1 + \cdots + a_n b_n| \leq (a_1^2 + \; \ldots + a_n^2)^{\frac{1}{2}} \, (b_1^2 + \; \ldots + b_n^2)^{\frac{1}{2}}$$

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \left(\sum_{i=1}^{n} y_i^2\right)$$

Proof:

Cauchy-Schwartz Inequality

Theorem

For two n-vectors $a \ and \ b$, $|a^T b| \leq \ ||a|| \ ||b||$

Written out:

$$|a_1 b_1 + \cdots + a_n b_n| \leq (a_1^2 + \ \ldots + a_n^2)^{\frac{1}{2}} (b_1^2 + \ \ldots + b_n^2)^{\frac{1}{2}}$$

$$2$$

$$(\sum_{i=1}^{n} x_i y_i) \leq (\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2)$$

It is clearly true if either $a \ or \ b \ is \ 0$.

So, assume $\alpha = ||a||$ and $\beta = ||b||$ are non-zero

We have

$$\begin{aligned} 0 &\leq \left|\left|\beta a - \alpha b\right|\right|^2 \\ &= \left|\left|\beta a\right|\right|^2 - 2\,(\beta a)^T(\alpha b) + \left|\left|\alpha b\right|\right|^2 \\ &= \beta^2 \left|\left|a\right|\right|^2 - 2\,\beta\alpha(a^T b) + \alpha^2 \left|\left|b\right|\right|^2 \\ &= 2\,\left|\left|a\right|\right|^2 \left|\left|b\right|\right|^2 - 2\,\left|\left|a\right|\right| \left|\left|b\right|\right| (a^T b) \end{aligned}$$

Divide by $2\,\left|\left|a\right|\right|\left|\left|b\right|\right|$ to get $a^T b \leq \left|\left|a\right|\right|\left|\left|b\right|\right|$

Apply to $-a, b$ to get other half of Cauchy-Schwartz inequality.

Cauchy-Schwarz inequality holds with equality when one of the vectors is a multiple of the other

If and only if a and b are linear dependent

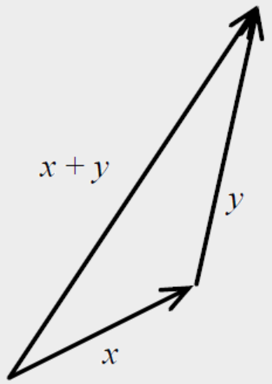
Triangle Inequality

Theorem

Consider a triangle in two or three dimensions:

$$||x + y|| \leq ||x|| + ||y||$$

Proof:

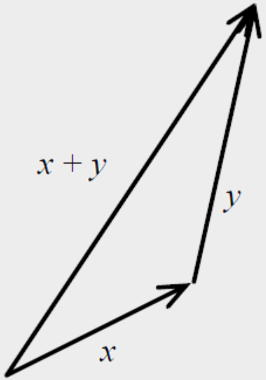


Triangle Inequality

Theorem

Consider a triangle in two or three dimensions:

$$||x + y|| \leq ||x|| + ||y||$$



Verification of triangle inequality:

$$||x + y||^2 = ||x||^2 + ||y||^2 + 2\, x^T y$$

$$\leq ||x||^2 + ||y||^2 + 2\, ||x||\,||y||$$

$$= (||x|| + ||y||)^2$$

$$\Rightarrow ||x + y|| \leq ||x|| + ||y||$$

Cauchy–Schwartz Inequality

Euclidean Norm

Vector Norm Properties

Important Properties:

1. Absolute Homogenity / Linearity:

$$||\alpha x|| = |\alpha| \, ||x||$$

2. Subadditivity / Triangle Inequality:

$$||x + y|| \leq ||x|| + ||y||$$

3. Positive definiteness / Point separating:

$$if \; ||x|| = 0 \; then \; x = 0$$
$$(from\ 1\ \&\ 3): For\ every\ x, ||x|| = 0\ iff\ x = 0$$

4. Non-negativity:

$$||x|| \geq 0$$

Norm of sum

Theorem

If x and y are vectors:

$$||x + y|| = \sqrt{||x||^2 + 2\, x^T y + ||y||^2}$$

Proof:

$$||x + y||^2 = (x + y)^T (x + y)$$

$$= x^T x + x^T y + y^T x + y^T y$$

$$= ||x||^2 + 2\, x^T y + ||y||^2$$

Inner product and norm

Theorem

Take any inner product $\langle \cdot, \cdot \rangle$ and define $f(x) = \sqrt{\langle x, x \rangle}$. Then f is a norm.

Proof

Note

Every inner product gives rise to a norm, but not every norm comes from an inner product. (Think about norm 2 and norm max)

Important

Suppose a,b,c are vectors:

$$\left\|\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right\|^2 = a^T a + b^T b + c^T c = ||a||^2 + ||b||^2 + ||c||^2$$

So, we have

$$\left\|\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right\| = \sqrt{||a||^2 + ||b||^2 + ||c||^2} = \left\|\begin{bmatrix} ||a|| \\ ||b|| \\ ||c|| \end{bmatrix}\right\|$$

(Parse RHS very carefully!)

❖ The norm of a stacked vector is the norm of the vector formed from the norms of sub-vectors.

Euclidean Metric (Distance)

Metric Properties

Important Properties:

Let V be a real vector space over $\mathbb{R}$. A function $V \times V \to \mathbb{R}$ is called **metric** or **distance function** on V, and (V, R) a metric space, if for all $u, v, w \in V$ the following holds true:

(i) $d(v, w) \geq 0$ and $d(v, w) = 0$ if and only if $v = w$;

(ii) $d(v, w) = d(v, w)$;

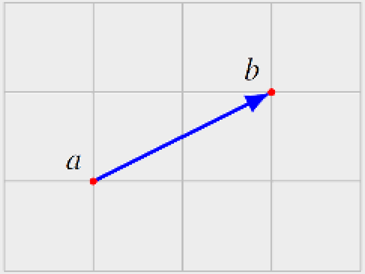
(iii) $d(v, w) \leq d(v, u) + d(u, w)$.

Euclidean Distance

Distance between two n−vectors shows the vectors are “close” or “nearby” or “far”.

Distance:

$$dist(a,b) = ||a - b||$$



Comparing Norm and Distance

Norm

(Normed Linear Space)

1. $||x-y|| \geq 0$
2. $||x-y|| = 0 \Rightarrow x = y$
3. $||\lambda(x-y)|| = |\lambda| \, ||x-y||$

Distance Function

(Metric Space)

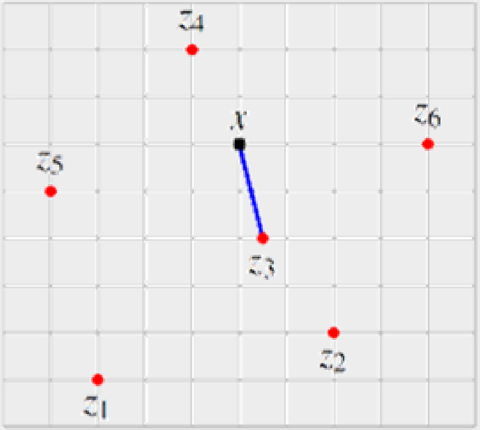
1. $dist(x, y) \geq 0$
2. $dist(x, y) = 0 \Rightarrow x = y$
3. $dist(x, y) = dist(y, x)$

Feature Distance and Nearest Neighbors:

$$if\ x, y\ are\ feature\ vectors\ for\ two\ entities, \left|\left|x - y\right|\right|\ is\ the\ feature\ distance$$

$if\ z_1, z_2, \dots, z_m\ is\ a\ list\ of\ vectors, z_j is\ the\ nearest\ neighbor\ of\ x\ if:$

$$\left|\left|x - z_j\right|\right| \leq \left|\left|x - z_i\right|\right|, \qquad i = 1, 2, \dots, m$$



Angle

Angle

Definition

Angle between two non−zero vectors a, b is defined as:

$$\angle(a, b) = \arccos\left(\frac{a^T b}{||a||||b||}\right)$$

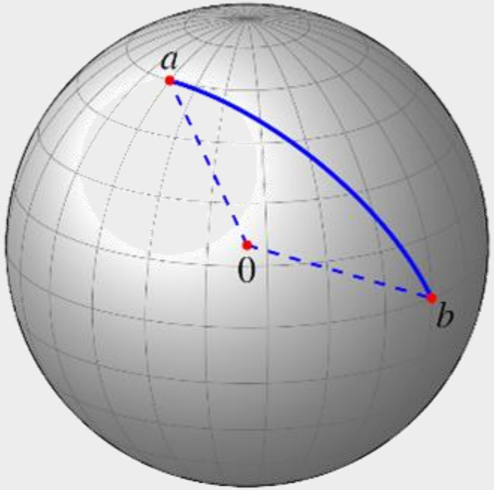
$\angle(a, b)$ is the number in $[0, \pi]$ that satisfies:

$$a^T b = ||a|| \, ||b|| \cos(\angle(a, b))$$

Coincides with ordinary angle between vectors in 2D and 3D

Spherical distance:

$$\text{if } a, b \text{ are on sphere with radius } R, \text{distance along the sphere is } R\,\angle(a, b)$$



References

- Chapter 1: Advanced Linear and Matrix Algebra, Nathaniel Johnston
- Chapter 6: Linear Algebra David Cherney
- Linear Algebra and Optimization for Machine Learning
- Introduction to Applied Linear Algebra Vectors, Matrices, and Least Squares